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Recurrent Proofs of the Irrationality of Certain Trigonometric Values

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In this note we exploit recurrences of integrals to give new elementary proofs of the irrationality of $\tan r$ for $r \in \mathbb{Q} \setminus \{0\}$ and $\cos r$ for $r^2 \in \mathbb{Q} \setminus \{0\}$. We also discuss applications of our technique to simpler irrationality proofs such as those for π, π^2, and certain values of exponential and hyperbolic functions.

1. IRRATIONALITY OF $\tan r$ FOR $r \in \mathbb{Q} \setminus \{0\}$. For a nonzero rational r, the irrationality of $\tan r$ was first proved by J. H. Lambert in 1761 by means of continued fractions [1, pp. 129–146]. We now present a new direct proof using a recurrence for an integral.

Theorem 1. *$\tan r$ is irrational for nonzero rational r.*

Proof. The irrationality of π will be a by-product of this proof, so we start by supposing that $r \in \mathbb{Q} \setminus \{k\pi : k \in \mathbb{Z}\}$. Write $r = a/b$ with $a, b \in \mathbb{Z}$ and assume that $\tan(r/2) = p/q$ with $p, q \in \mathbb{Z}$. For $n \geq 0$, let $f_n(x) = (rx - x^2)^n/n!$ and $I_n = \int_0^r f_n(x) \sin x \, dx$. Then $b^n I_n \to 0$ as $n \to \infty$, $I_0 = 1 - \cos r$, and $I_1 = 2(1 - \cos r) - r \sin r$. Integrating by parts twice, we get that for $n \geq 2$,

$$I_n = -\int_0^r f_n''(x) \sin x \, dx = (4n - 2)I_{n-1} - r^2 I_{n-2}. \tag{1}$$

By inducting on n using (1), we see that for $n \geq 0$, $I_n = u_n(1 - \cos r) + v_n \sin r$, where u_n and v_n are polynomials in r with integer coefficients and degrees at most n. Moreover, if two consecutive terms of the sequence $\langle I_n \rangle$ are 0, then (1) forces all terms of $\langle I_n \rangle$ to be 0, and in particular $I_0 = 0$, a contradiction. Hence $\langle I_n \rangle$ has infinitely

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many nonzero terms. Therefore, we can pick a large enough n so that $b^n q \csc r I_n = b^n q[u_n \tan(r/2) + v_n]$ is a nonzero integer in $(-1, 1)$, a contradiction.

Notice that $\tan(\pi/4) = 1$, so $\pi/2 \notin \mathbb{Q}$, which implies that $\mathbb{Q} \setminus \{k\pi : k \in \mathbb{Z}\} = \mathbb{Q} \setminus \{0\}$. Thus we have proved that $\tan(r/2) \notin \mathbb{Q}$ for all $r \in \mathbb{Q} \setminus \{0\}$. ∎

A closer inspection of our proof reveals that u_n and v_n/r are polynomials in r^2. So a slightly stronger conclusion can be squeezed out of the proof, namely that $(\tan r)/r$ is irrational whenever $r^2 \in \mathbb{Q} \setminus \{0\}$. This stronger result was first established through a different elementary approach by Inkeri [2].

2. SIMPLER PROOFS OF IRRATIONALITY. From the previous proof we see that recurrence is a double-edged sword. It is sharp and swift in showing that a sequence is integer-valued and has an infinite nonzero subsequence. We can also use recurrence to give similar proofs of the irrationality of π, π^2, e^r, etc. However, in these easier cases, the corresponding sequences of integrals are positive, so there is no need to argue for the existence of a nonzero subsequence. Consequently the proofs can be really short and charming. For a flavor of it, we present a direct proof of the irrationality of π which is even shorter than the celebrated one-page proof given by Niven [3].

Theorem 2. *π is irrational.*

Proof. Assume that $\pi = a/b$ with $a, b \in \mathbb{N}$. Let $f_n(x) = (\pi x - x^2)^n/n!$ and $I_n = \int_0^\pi f_n(x) \sin x \, dx$. Then $b^n I_n \to 0$ as $n \to \infty$, $I_0 = 2$, and $I_1 = 4$. Replacing r by π in (1) we get $I_n = (4n-2)I_{n-1} - \pi^2 I_{n-2}$ for $n \geq 2$. By induction on n using this recurrence, we see that for $n \geq 0$, I_n is a polynomial in π with integer coefficients and degree at most n. Hence for a large enough n, $b^n I_n$ is an integer in $(0, 1)$, a contradiction. ∎

Notice that the terms of $\langle I_n \rangle$ are really polynomials in π^2, so our proof only needs very minor changes to show the stronger conclusion that π^2 is irrational. In fact in [5], Schröder presented a very similar proof by recurrence of the irrationality of π^2.

For the irrationality of e^r for nonzero rational r, the interested reader can imitate the process with the sequence $I_n = \int_0^r f_n(x) e^x \, dx$, where $f_n(x) = (rx - x^2)^n/n!$ as well.

3. IRRATIONALITY OF $\cos r$ FOR $r^2 \in \mathbb{Q} \setminus \{0\}$. Next we turn our attention to the cosine function. The classical elementary proof of the irrationality of $\cos r$ for nonzero rational r was given by Niven [4, Theorem 2.5, pp. 16–19]. Niven's proof can be slightly modified to show that $\cos r$ is irrational whenever $r^2 \in \mathbb{Q} \setminus \{0\}$, as observed by Inkeri [2]. We now use the full force of recurrence to give a proof which is more direct than Inkeri's modification.

Theorem 3. *If $r^2 \in \mathbb{Q} \setminus \{0\}$ then $\cos r$ is irrational.*

Proof. Assume that $r^2 = a/b$ with $a, b \in \mathbb{Z} \setminus \{0\}$ and $\cos r = p/q$ with $p, q \in \mathbb{Z}$. For $n \geq 0$, let $f_n(z) = (r^2 z^2 - z^4)^n/n!$, $I_n = \int_0^r f_n(z) \sin(r - z) \, dz$, $J_n = \int_0^r z f_n(z) \cos(r - z) \, dz$, $K_n = \int_0^r z^2 f_n(z) \sin(r - z) \, dz$, and $L_n = \int_0^r z^3 f_n(z) \cos(r - z) \, dz$. Then b^{2n+1} times each of the four integrals approaches 0 as $n \to \infty$. Direct integration yields $I_0 = 1 - \cos r = J_0$, $K_0 = r^2 - 2 + 2\cos r$, and $L_0 = 3K_0$. Integrating each integral by parts once, we get that for $n \geq 1$,

$$I_n = 4L_{n-1} - 2r^2 J_{n-1}, \tag{2}$$

$$J_n = (4n+1)I_n - 2r^2 K_{n-1}, \tag{3}$$

$$K_n = -(4n+2)J_n + 2r^2 L_{n-1}, \tag{4}$$

$$L_n = (4n+3)K_n + 2nr^2 I_n - 2r^4 K_{n-1}. \tag{5}$$

Induction on n in these recurrences implies that for $n \geq 0$, the four sequences have the form $u_n + v_n \cos r$, where u_n and v_n are polynomials in r^2 with integer coefficients and degrees at most $2n+1$. Moreover, suppose that $I_m = J_m = K_m = L_m = 0$ for some $m \geq 1$. Then (3) and (4) imply that $K_{m-1} = 0$ and $L_{m-1} = 0$. Thus (2) yields $J_{m-1} = 0$. Eliminating K_{n-1} from (3) and (5), we see that for $n \geq 1$, I_n can be expressed in terms of J_n, K_n, and L_n; and keep in mind also that $I_0 = J_0$. Hence $I_{m-1} = 0$. By this argument of infinite descent we conclude that $I_0 = J_0 = K_0 = L_0 = 0$, which contradicts the fact that $2I_0 + K_0 = r^2 \neq 0$. Therefore, at least one of the four sequences has infinitely many nonzero terms. Pick such a sequence and a large enough n so that $b^{2n+1}q$ times the corresponding integral is a nonzero number in $(-1, 1)$, while it also has the form $b^{2n+1}q(u_n + v_n \cos r)$, which is an integer, a contradiction. Thus $\cos r \notin \mathbb{Q}$ whenever $r^2 \in \mathbb{Q} \setminus \{0\}$. ■

Corollary 1. *π^2 is irrational. Also, if $r^2 \in \mathbb{Q} \setminus \{0\}$ then $\sin^2 r$, $\cos^2 r$, and $\tan^2 r$ are all irrational.*

Proof. The claims follow immediately from Theorem 3 and the identities $\cos \pi = -1$ and $\cos 2r = 1 - 2\sin^2 r = 2\cos^2 r - 1 = (1 - \tan^2 r)/(1 + \tan^2 r)$. ■

Finally, the observant reader perhaps has noticed that our proof of Theorem 3 allows r^2 to be a negative rational number. Since $\cosh r = \cos(ir)$, all the analogous statements about hyperbolic functions are included in our results. The skeptical reader is invited to work out the details by substituting $r = it$ and $z = iy$ into I_n, J_n, K_n, and L_n. In fact, the resulting real integrals are nonzero and thus the proof is shorter, because the argument of infinite descent is not needed.

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